W3L2 - CAUCHY-EULER DIFFERENTIAL EQUATIONS

Form: $ax^2y'' + bxy' + cy = g(x)$

Homogeneous when $g(x) = 0$
Non-Homogeneous when $g(x) \neq 0$

Overall Method: Find solutions of the form $y = x^m$

$ax^2y'' + bxy' + cy = 0$
Homogeneous form: Assume $y = x^m$
$y' = mx^{m-1}$
$y'' = m(m-1)x^{m-2}$

$$ax^2\left[m(m-1)x^{m-2}\right] + bx\left[mx^{m-1}\right] + c\left[x^m\right] = 0$$

$$ax^2\left[m(m-1)\right] + bx^m\left[m\right] + c\left[x^m\right] = 0$$

$$x^m\left[am(m-1) + bm + c\right] = 0$$

$$x^m\left[am^2 - am + bm + c\right] = 0$$

$$x^m\underline{\left[am^2 + (b-a)m + c\right]} = 0$$

x^m is a solution whenever this expression is equal to zero

$ax^2y'' + bxy' + cy = 0$
Homogeneous form - Distinct Real Roots

$x^2y'' + 7xy' + 8y = 0$ ← $a = 1$
$b = 7$
$c = 8$

$am^2 + (b-a)m + c = 0$
$m^2 + 6m + 8 = 0$
$(m+4)(m+2) = 0$

$\underline{y = C_1x^{-2} + C_2x^{-4}}$ $y = c_1x^{m_1} + c_2x^{m_2}$

$ax^2y'' + bxy' + cy = 0$
Homogeneous form - Repeated Real Roots

$m_1 = 3,\ m_2 = 3 \rightarrow y = C_1e^{3x} + C_2xe^{3x}$ $y = c_1x^m + c_2x^m \ln x$

$9x^2y'' + 3xy' + y = 0$ → $a = 9$
$am^2 + (b-a)m + c = 0$ $b = 3$
$c = 1$

$9m^2 - 6m + 1 = 0$
$(3m-1)(3m-1) = 0$
$m_1 = \frac{1}{3}$ $m_2 = \frac{1}{3}$

$\underline{y = c_1x^{\frac{1}{3}} + c_2x^{\frac{1}{3}}\ln(x)}$

$ax^2y'' + bxy' + cy = 0$

Homogeneous form - Complex Roots

$m = \alpha \pm \beta i \rightarrow y = e^{x}[c_1 \cos \beta x + c_2 \sin \beta x]$

$y = x^{\alpha}[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$

$x^2y'' - 9xy' + 28y = 0 \rightarrow a = 1$

$am^2 + (b-a)m + c = 0$

$b = -9$

$c = 28$

$m^2 - 10m + 28 = 0$

$m = 5 \pm \sqrt{3}\, i$

$\alpha = 5 \quad \beta = \sqrt{3}$

$\underline{y = x^5[c_1 \cos(\sqrt{3} \ln x) + c_2 \sin(\sqrt{3} \ln x)]}$

Non-homogeneous form solution method

1. Solve the associated homogeneous equation (this gives y_c)
2. Divide equation by ax^2 to put the equation in standard form
3. Use "Variation of Parameters" method to find y_p
4. Solution is $y = y_c + y_p$ (Be sure all terms linearly independent)